Fiberwise compactifications of Lefschetz fibrations

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1 Geometry of Lefschetz fibrations

1.1 Anticanonical Lefschetz pencil

Let

$$\bar{\pi} : \overline{E} \to \mathbb{C} \tag{1}$$

be a symplectic Lefschetz fibration with closed fibers. We require that at any regular point of a fiber, the restriction of $\omega_{\overline{E}}$ to the symplectic orthogonal complement of the fiberwise tangent space is positive (to ensure that symplectic parallel transport is well-defined). Assume that $c_1(\overline{E}) = 0$, so that the smooth fiber $\overline{M}$ also inherits a Calabi-Yau structure. Write $2n = \dim(\overline{E}) \geq 4$.

One particular class of such Lefschetz fibrations arises from a Lefschetz fibration

$$\overline{\pi}\rceil : \overline{E}\rceil \to \mathbb{CP}^1. \tag{2}$$

We assume that $c_1(\overline{E}\rceil) = PD([\overline{M}])$, and in fact that $\overline{E}\rceil$ comes with a preferred homotopy class of isomorphisms

$$\mathcal{K}_{\overline{E}\rceil} \cong \overline{\pi}\rceil^* \mathcal{O}_{\mathbb{P}^1}(-1). \tag{3}$$

We also assume that the fiber at infinity is smooth, so that we can take it to be $\overline{M}$. Setting $\overline{E} = \overline{E}\rceil \setminus \overline{M}$ and restricting $\overline{\pi}\rceil$ yields a Lefschetz fibration $\bar{\pi}$.

We will work with Lefschetz fibrations relative to a fiberwise ample divisor. By a *fiberwise divisor* we mean a properly embedded symplectic hypersurface

$$\delta E \subset \overline{E}, \tag{4}$$

such that

- no critical points of $\bar{\pi}$ line over δE;
- the symplectic parallel transport vector fields are tangent to δE, and give rise to a flat connection for $\delta E \to \mathbb{C}$.

Denote by $\delta M \subset \overline{M}$ the restriction of $\delta \overline{E}$ to the fiber M, then $\delta E \cong \mathbb{C} \times \delta M$ by parallel transport. The ampleness assumption says that

$$[\omega_{\overline{E}}] = [\delta E] \in H^2(\overline{E}). \tag{5}$$

By restricting $\bar{\pi}$ to $E = \overline{E} \setminus \delta E$, we get a Lefschetz fibration

$$\pi : E \to \mathbb{C}, \tag{6}$$

whose fiber $M = \overline{M} \setminus \delta M$ is a Liouville manifold. We say that $\bar{\pi}$ is a fiberwise compactification of π.

There is a counterpart for the Lefschetz fibratiion $\overline{\pi}\rceil$, where we assume the existence of a fiberwise divisor

$$\delta E\rceil \subset \overline{E}\rceil. \tag{7}$$

There are three conditions on this:

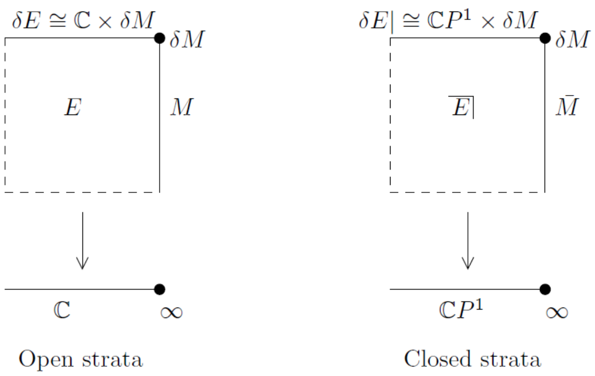

Figure 1: Illustration of the basic geometric set up

- parallel transport should be tangent to $\delta E|$, and gives rise to a flat connection for $\delta E| \to \mathbb{CP}^1$, hence a preferred identification $\delta E| \cong \mathbb{CP}^1 \times \delta M$;
- with respect to the above identification, the normal bundles to $\delta E| \subset \overline{E}|$ and $\delta M \subset \overline{M}$ must be related by
$$\nu_{\delta E|} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \nu_{\delta M}; \tag{8}$$
-
$$[\omega_{\overline{E}|}] = [\delta E|] + \lambda[\overline{M}] \in H^2(\overline{E}|) \tag{9}$$
for some $\lambda \in \mathbb{R}$ (it would be important later that one could take λ to be arbitrarily large).

In this setup, we will say that π and its fiberwise compactification $\bar{\pi}$ *arise from an anticanonical Lefschetz pencil.* One can blow down $\delta E| \subset \overline{E}|$ along the $\mathbb{CP}^1$ fibers. The outcome is a symplectic $2n$-manifold $\overline{E}\rceil$ carrying a Lefschetz pencil of hypersurfaces isomorphic to $\overline{M}$, with base locus δM. $c_1(\overline{E}\rceil) = PD([\overline{M}])$, which is also a positive multiple of the symplectic class. In particular, $\overline{E}\rceil$ is monotone.

1.2 Fukaya categories of Lefschetz fibrations

We will work over $\mathbb{C}$, and denote the Fukaya category associated to the Lefschetz fibration π by $\mathcal{F}(\pi)$. By our assumption, $\mathcal{F}(\pi)$ is a $\mathbb{Z}$-graded A_∞-category. A fibrewise compactification gives rise to a formal deformation $\mathcal{F}_q(\bar{\pi})$ of $\mathcal{F}(\pi)$, which is still $\mathbb{Z}$-graded, and is defined over $\mathbb{C}[[q]]$. This is a version of the relative Fukaya category (relative to the fiberwise ample divisor δE); the formal variable q counts the intersection number of pseudo-holomorphic maps with δE.

There are twisted versions $\mathcal{F}_b(\pi)$ of $\mathcal{F}(\pi)$, associated to a choice of bulk term
$$b \in H^2(E; \mathbb{C}^*). \tag{10}$$
Roughly speaking, for $A \in H_2(E)$, $b \cdot A \in \mathbb{C}^*$ is a weight with which pseudo-holomorphic curves in class A are counted. Similarly, there are twisted versions $\mathcal{F}_{q,\bar{b}}(\bar{\pi})$ of $\mathcal{F}_q(\bar{\pi})$, where
$$\bar{b} \in H^2(\overline{E}; \mathbb{C}[[q]]^\times). \tag{11}$$
The choices of b and $\bar{b}$ are related as follows: $\mathcal{F}_{q,\bar{b}}(\bar{\pi})$ is a formal deformation of $\mathcal{F}_b(\pi)$, where b is obtained from $\bar{b}$ by setting $q = 0$ and then restricting to E. Again, the bulk term determines weights of holomorphic curves, which are now given by
$$q^{\delta E \cdot A}(\bar{b} \cdot A) \tag{12}$$

(the first factor $q^{\delta E \cdot A}$ above appears already in the definition of $\mathcal{F}_q(\bar{\pi})$). Rerpesentations of q have a natural interpretation within this framework. Namely, suppose that we take $\mathcal{F}_{q,\bar{b}}(\bar{\pi})$ and apply a parameter change

$$q \mapsto \beta(q), \beta(0) = 0, \beta'(0) \neq 0. \tag{13}$$

The outcome is isomorphic to $\mathcal{F}_{q,\bar{b}_\beta}(\bar{\pi})$, where

$$\bar{b}_\beta(q) = \bar{b}(\beta(q)) \cdot (\beta(q)/q)^{[\delta E]}. \tag{14}$$

The first term in (14) is obtained by applying the parameter change to the multiplicative group $\mathbb{C}[[q]]^\times$ in which $\bar{b}$ takes its values; the second term maps $A \in H_2(\overline{E})$ to $(\beta(q)/q)^{\delta E \cdot A} \in \mathbb{C}[[q]]^\times$. In particular, if $H^2(\overline{E}) \cong \mathbb{C}\langle \delta E \rangle$, then any $\mathcal{F}_{q,\bar{b}}(\bar{\pi})$ is just a reparametrized version of $\mathcal{F}_q(\bar{\pi})$ (geometrically, one can understand this as a version of the fact that Fukaya category is invariant up to quasi-isomorphism under the rescaling of the symplectic form). It's helpful to think of q as the variable in a symplectic class

$$[\omega_{\overline{E},q,\bar{b}}] = -\log(q)[\delta E] - \log(\bar{b}). \tag{15}$$

Formally, (12) can be written in the more familiar form

$$q^{\delta E \cdot A}(\bar{b} \cdot A) = \exp\left(-\int_A \omega_{\overline{E},q,\bar{b}}\right). \tag{16}$$

Taking the derivative of (15) gives

$$-\partial_q[\omega_{\overline{E},q,\bar{b}}] = q^{-1}[\delta E] + (\partial_q \bar{b})/\bar{b} \in H^2(\overline{E}) \otimes q^{-1}\mathbb{C}[[q]]. \tag{17}$$

We will assume throughout the rest of the paper that b is trivial, and correspondingly specialize $\bar{b}$ to

$$\bar{b} \in H^2(\overline{E}; 1 + q\mathbb{C}[[q]]) \subset H^2(\overline{E}, \mathbb{C}[[q]]^\times). \tag{18}$$

(The reason for using this particular choice lies in Assumption 2.1, which shows that we can already find a solution to this deformation problem within this subspace of $H^2(\overline{E}, \mathbb{C}[[q]]^\times)$.) As a consequence, only parameter changes with $\beta'(0) = 1$ will be allowed.

Instead of working with the entire Fukaya categories associated to Lefschetz fibrations, we fix a basis of Lefschetz thimbles $\{L_1, \cdots, L_m\}$ and consider the resulting full A_∞-subcategories

$$\mathcal{A} \subset \mathcal{F}(\pi),\ \mathcal{A}_{q,\bar{b}} \subset \mathcal{F}_{q,\bar{b}}(\bar{\pi}). \tag{19}$$

It's known that the embedding $\mathcal{A} \subset \mathcal{F}(\pi)$ induces an equivalence of derived categories. As a consequence, the restriction map on Hochschild cohomology

$$HH^*(\mathcal{F}(\pi), \mathcal{F}(\pi)) \to HH^*(\mathcal{A}, \mathcal{A}) \tag{20}$$

is an isomorphism. This implies that the deformation $\mathcal{F}_{q,\bar{b}}(\bar{\pi})$ is determined up to isomorphism by $\mathcal{A}_{q,\bar{b}}$.

However, it's usually true that if one restricts to a full A_∞-subcategory $\mathcal{T} \subset \mathcal{F}(\pi)$, then the deformation of $\mathcal{F}(\pi)$ is controlled by that of $\mathcal{T}$. Namely, there is an injective map

$$HH^*(\mathcal{F}(\pi), \mathcal{F}(\pi)) \hookrightarrow HH^*(\mathcal{T}, \mathcal{T}). \tag{21}$$

1.3 Fukaya categories of fibers

Our basis of Lefschetz thimbles gives rise to a collection of vanishing cycles $V_1, \cdots, V_m \subset M$. Let

$$\mathcal{B} \subset \mathcal{F}(M) \tag{22}$$

be the full A_∞-subcategory formed by these cycles. Up to quasi-isomorphism, $\mathcal{A}$ can be identified with the directed subcategory of $\mathcal{B}$.

A similar relationship holds after fiberwise compactification. Consider the analogue of $\mathcal{B}$ inside the relative Fukaya category of $(\overline{M}, \delta M)$, twisted by the restriction of $\bar{b}$:

$$\mathcal{B}_{q,\bar{b}|\overline{M}} \subset \mathcal{F}_{q,\bar{b}|\overline{M}}(\overline{M}). \tag{23}$$

Lemma 1.1. *$\mathcal{B}_{q,\bar{b}|\overline{M}}$ has zero curvature, and $\mathcal{A}_{q,\bar{b}}$ is quasi-isomorphic to its directed A_∞-subcategory.*

Strictly speaking, $\mathcal{B}_{q,\bar{b}|\overline{M}}$ is well-defined only up to quasi-isomorphism (of deformations of $\mathcal{B}$). The statement says that there are representatives with vanishing μ^0. For $n \neq 2$, the existence of such a representative is clear from homological algebra. For $n = 2$, it follows from index and dimension reasons. (Note that Lemma 1.1 is the main reason for restricting our attension to vanishing cycles and Lefschetz thimbles, as $\mathcal{F}_{q,\bar{b}|\overline{M}}$ is generally curved.)

Remark. In the case where the Lefschetz fibration comes from an anticanonical Lefschetz pencil, it's known that $\mathcal{B}$ spit-generates $\mathcal{F}(M)$. (In fact, the same is true for Lefschetz pencils whose fibers represent an integral multiple of $c_1(\overline{E}|)$. The proof is a combinations of Seidel's long exact sequence together with the fact that the Reeb flow on ∂M is a circle action.) The analogous question for $\mathcal{B}_{q,\bar{b}|\overline{M}}$ comes in several different versions. To take one of them, let

$$\overline{\mathbb{C}((q))} \supset \mathbb{C}((q)) \tag{24}$$

be the algebraic closure of $\mathbb{C}((q))$, obtained by adjoining roots $q^{1/d}$ of arbitrary order. There is a version of the Fukaya category of $\overline{M}$ (no longer relative to δM) defined over $\overline{\mathbb{C}((q))}$, and a $\bar{b}|\overline{M}$-twisted generalization. Then, that category is split-generated by its full A_∞-subcategory $\mathcal{B}_{q,\bar{b}} \otimes_{\mathbb{C}[[q]]} \overline{\mathbb{C}((q))}$. (Another approach, without referring to Seidel's long exact sequence, is Ganatra-Perutz-Sheridan's automatic split-generation.)

1.4 Gromov-Witten invariants

For $A \in H_2(\overline{E}|;\mathbb{Z})$ and $d \geq 0$, consider the Gromov-Witten invariant counting genus 0 curves with d marked points in class A, which is a map

$$\langle \cdots \rangle_A : H^*(\overline{E}|)^{\otimes d} \to \mathbb{C}. \tag{25}$$

The number $\langle x_1, \cdots, x_d \rangle$ is non-zero only if the degrees satisfy

$$|x_1| + \cdots + |x_d| = 2n + 2(d-3) + 2\overline{M} \cdot A. \tag{26}$$

The contribution of constant maps $(A = 0)$ is

$$\langle x_1, \cdots, x_d \rangle_0 = \begin{cases} \int_{\overline{E}|} x_1 x_2 x_3 & d = 3 \\ 0 & d \neq 3 \end{cases} \tag{27}$$

The *fundamental class axiom* and *divisor axiom* say that for $A \neq 0$,

$$\langle 1, x_2, \cdots, x_d \rangle_A = 0, \tag{28}$$

$$\langle x_1, \cdots, x_d \rangle_A = (x_1 \cdot A)\langle x_2, \cdots, x_d \rangle_A, |x_1| = 2. \tag{29}$$

Since (25) is defined by counting stable maps representing A, it's zero unless either $A = 0$ or $\int_A \omega_{\overline{E}|} > 0$. This is sometimes called *effectiveness axiom*. Note that by deforming the symplectic form, one can make the constant λ in $\omega_{\overline{E}|}$ arbitrarily large. Hence, (recall that $[\omega_{\overline{E}|}] = [\delta E|] + \lambda[\overline{M}]$)

$$\langle \cdots \rangle = 0 \text{ if either } \begin{cases} \overline{M} \cdot A < 0 \\ \overline{M} \cdot A = 0, \delta E| \cdot A \leq 0, A \neq 0 \end{cases} \tag{30}$$

The final statement we need is more specific to our geometric situation:

$$\textit{If } \delta E| \cdot A < 0, \ \textit{then } \langle \cdots \rangle_A = 0 \ \textit{unless } A \textit{ lies in the image of } H_2(\delta E|) \to H_2(\overline{E}|). \tag{31}$$

This is a property of Gromov-Witten invariants of blowups.

When summing over classes A, we use a formal parameter q as before, and allow a bulk term

$$\overline{b}| \in H^2(\overline{E}|; 1 + q\mathbb{C}[[q]]) \subset H^2(\overline{E}|; \mathbb{C}[[q]]^\times). \tag{32}$$

Concretely, this means that

$$\langle x_1, \cdots, x_d \rangle_{q,\overline{b}|} = \sum_A q^{\delta E|\cdot A}(\overline{b}| \cdot A)\langle x_1, \cdots, x_d\rangle_A \in \mathbb{C}((q)). \tag{33}$$

(Since the possibility of $\delta E| \cdot A < 0$ is not excluded, we need to use the Laurent power series ring in the above.) One can formally write the weights in (33) as

$$q^{\delta E|\cdot A}(\overline{b}| \cdot A) = \exp\left(-\int_A \omega_{\overline{E}|,q,\overline{b}|}\right), \tag{34}$$

where

$$[\omega_{\overline{E}|,q,\overline{b}|}] = -\log(q)[\delta E|] - \log(\overline{b}|) \tag{35}$$

and hence

$$-\partial_q[\omega_{\overline{E}|,q,\overline{b}|}] = q^{-1}[\delta E|] + (\partial_q \overline{b}|)/\overline{b}| \in H^2(\overline{E}|) \otimes q^{-1}\mathbb{C}[[q]]. \tag{36}$$

The fact that $[\delta E|] - [\omega_{\overline{E}|}]$ is a multiple of $c_1(\overline{E}|) > 0$ ensures the q-adic convergence of (33). As a consequence of the divisor axiom, we have

$$\langle \partial_q[\omega_{\overline{E}|,q,\overline{b}|}], x_1, \cdots, x_d\rangle_{q,\overline{b}|} + \partial_q\langle x_1, \cdots, x_d\rangle_{q,\overline{b}|} = 0 \textit{ for } d \neq 2. \tag{37}$$

The class $[\overline{M}]$ also plays a distinguished role: if I is the grading operator defined by

$$I(x) = \frac{|x|}{2}x, \tag{38}$$

then

$$\begin{aligned} &\langle [\bar{M}], x_1, \cdots, x_d\rangle_{q,\overline{b}|} + (n + d - 3)\langle x_1, \cdots, x_d\rangle_{q,\overline{b}|} \\ &\quad = \langle Ix_1, \cdots, x_d\rangle_{q,\overline{b}|} + \cdots + \langle x_1, \ldots, Ix_d\rangle_{q,\overline{b}|} \quad \text{for } d \neq 2. \end{aligned} \tag{39}$$

In both (37) and (39), $d = 2$ is excluded because of the contribution of constant maps to the left hand side. Usually, we will rewrite the Gromov-Witten invariants (Poincaré dually) as multilinear maps

$$z_{q,\overline{b}|} : H^*(\overline{E}|)((q))^{\otimes d-1} \to H^*(\overline{E}|)((q)). \tag{40}$$

We further break them up into graded pieces $z^{(k)}_{q,\overline{b}|}$, by restricting the summation to classes A with $\overline{M} \cdot A = k$.

In the simplest case $d = 1$, we get a class

$$z_{q,\overline{b}|} \in H^*(\overline{E}|)((q)). \tag{41}$$

Because of (30), this has only three nontrivial graded components, which have the following more precise form:

$$z^{(0)}_{q,\overline{b}|} \in H^4(\overline{E}|) \otimes q\mathbb{C}[[q]], \tag{42}$$

$$z^{(1)}_{q,\overline{b}|} \in q^{-1}[\delta E|] + H^2(\overline{E}|)[[q]], \tag{43}$$

$$z^{(2)}_{q,\overline{b}|} \in H^0(\overline{E}|; \mathbb{C}[[q]]). \tag{44}$$

Geometrically, $z^{(0)}_{q,\overline{b}|}$ counts pseudoholomorphic curves in the fibers. Again due to (30), only positive powers of q appear in it. Next, $z^{(1)}_{q,\overline{b}|}$ counts sections. Due to (31), the only classes A that can contribute with negative powers of q come from $H_2(\delta E|) \cong H_2(\mathbb{CP}^1) \oplus H_2(\delta M)$. Using (8), one sees that the only such contribution comes from the generator of $H_2(\mathbb{CP}^1)$, which geometrically means from trivial sections

$$\mathbb{CP}^1 \times \{pt\} \subset \mathbb{CP}^1 \times \delta M = \delta E| \subset \overline{E}|. \tag{45}$$

This explains the form of the leading order term in (43). Finally, $z^{(2)}_{q,\overline{b}|}$ counts bisections, and a similar argument shows that no negative powers of q appear in it.

The special case $d = 3$ of (40) defines the small quantum product, usually written as

$$x_1 * x_2 = z_{q,\overline{b}|}(x_1, x_2). \tag{46}$$

This is associative. More specifically for our situation, the divisor axiom implies that

$$[\overline{M}] * [\overline{M}] = z^{(1)}_{q,\overline{b}|} + 4z^{(2)}_{q,\overline{b}|}. \tag{47}$$

There is a relation between the quantum product structure and counting holomorphic discs with Lagrangian boundary conditions. Let $L \subset E$ be a graded, *Spin*, closed, exact Lagrangian submanifold. This defines an object of $\mathcal{F}(E) \subset \mathcal{F}(\pi)$. Suppose that L admits a deformation to an unobstructed object of the Fukaya category $\mathcal{F}(\overline{E}, \delta E)_{q,\bar{b}} \subset \mathcal{F}_{q,\bar{b}}(\bar{\pi})$. Then, by counting holomorphic discs with boundary on L which go exactly once through the fiber $\overline{M} \subset \overline{E}|$ at infinity, we get an invariant

$$W_{L,q,\overline{b}|} \in H^0(\hom_{\mathcal{F}_{q,\bar{b}}}(\bar{\pi})(L,L)) = HF^0_{\overline{E},\bar{b}}(L,L) = \mathbb{C}[[q]]. \tag{48}$$

The constant (q^0) term counts Maslov index 2 discs in $(\overline{E}|, L)$. One can view $W_{L,q,\overline{b}|}$ as an extension of $\mathfrak{m}_0(L)$ in the monotone Fukaya category $\mathcal{F}(\overline{E}|)$, which includes higher Maslov index holomorphic discs in $\overline{E}|$ with tangency conditions to the base locus of the pencil. The class $[L] \in H^n(\overline{E}|)$ satisfies

$$[\overline{M}] * [L] = W_{L,q,\overline{b}|}[L], \tag{49}$$

$$(\partial_q[\omega_{\overline{E}|,q,\overline{b}|}]) * [L] + (\partial_q W_{L,q,\overline{b}|})[L] = 0. \tag{50}$$

These statements are consequences of the formalism of open-closed string maps.

In particular, the first one is the generalization of the fact that

$$CO^0(c_1(\overline{E}|)) = \mathfrak{m}_0(L)[L] \tag{51}$$

in the monotone Fukaya category $\mathcal{F}(\overline{E}|)$, where $CO^0 : QH^*(\overline{E}|) \to HF^*(L,L)$ is the degree 0 part of the closed-open map.

1.5 Fixed point Floer cohomology

This is a Floer cohomology associated to general symplectic automorphisms, which generalizes Hamiltonian Floer cohomlogy. Let M be a Liouville domain, which should be thought of as a fiber of the Lefschetz fibration π in our particular set up. We will consider the following situation:

(i) Let ϕ be a symplectic automorphism of M which is equal to the identity near ∂M and exact. The latter condition means that there is a function G_ϕ such that $\phi^*\theta_M - \theta_M = dG_\phi$.

(ii) If M carries a symplectic Calabi-Yau structure, we will assume that ϕ is compatible with it, and in fact comes with a choice of grading, making it a graded symplectic automorphism.

Take families of functions $H = (H_t)$ and $J = (J_t)$, parametrized by $t \in \mathbb{R}$. We require that for every $H_t \in C^\infty(M, \mathbb{R})$, $H = \varepsilon\rho_M$ near ∂M for some $\varepsilon \in \mathbb{R}$, where ρ_M satisfies

$$\begin{cases} \rho_M|\partial M = 1, \\ Z_M \cdot \rho_M = \rho_M, \end{cases} \tag{52}$$

with Z_M being the Liouville vector field; and that for every J_t, $\theta_M \circ J_t = d\rho_M$ near ∂M. Instead of using 1-periodic Hamiltonians and almost complex structures, we now consider their ϕ-twisted versions:

$$\begin{cases} H_{t+1}(x) = H_t(\phi(x)), \\ J_{t+1} = \phi^* J_t. \end{cases} \tag{53}$$

Take the twisted loop space $\mathcal{L}_\phi = \{x : \mathbb{R} \to M : x(t) = \phi(x(t+1))\}$, with action functional

$$A_{\phi,H}(x) = \left(\int_0^1 -x^*\theta_M + H_t(x(t))dt \right) - G_\phi(x(1)). \tag{54}$$

It critical points are solutions $x \in \mathcal{L}_\phi$ of

$$\frac{dx}{dt} = X_{H_t}, \tag{55}$$

and correspond bijectively to fixed points $x(1)$ of $\phi_H^1 \circ \phi$. For a generic choice of H, these will be nondegenerate, and we use them as generators of a $\mathbb{Z}/2$-graded $\mathbb{K}$-vector space $CF^*(\phi, H)$. To define the differential, one considers solutions $u : \mathbb{R}^2 \to M$ of

$$\partial_s u + J_t(\partial_t u - X_{H_t}) = 0, \tag{56}$$

which satisfy

$$u(s,t) = \phi(u(s, t+1)). \tag{57}$$

The resulting Floer cohomology is denoted by $HF^*(\phi, \varepsilon)$. When the condition (ii) above is satisfied for M and ϕ, one can define a $\mathbb{Z}$-grading on $HF^*(\phi, \varepsilon)$.

Example. Suppose that ∂M is a contact circle bundle. Fix a function F on M which agrees with ρ_M near the boundary, and let (ϕ_F^t) be its flow. By construction, ϕ_F^{-1} is the identity near the boundary. We call it the *boundary twist* of M, and denote it by $\tau_{\partial M}$. (In the case of an anticanonical Lefschetz fibration, $\tau_{\partial M}$ is nothing else but a composition of Dehn twists along vanishing cycles, which is in particular supported away from ∂M.)

Let's now assume that M carries a symplectic Calabi-Yau structure. Then, $\tau_{\partial M}$ can be equipped with the structure of a graded symplectic automorphism. In fact, there are two reasonable choices (which coincide for $m = 1$):

(i) One can take the trivial grading of the identity, and extend it continuously over the isotopy (ϕ_F^t) to get a grading of $\tau_{\partial M}$. Near the boundary, the grading is a shift $[2 - 2m]$.

(ii) Alternatively, by changing the previous grading by a constant $2 - 2m$, one can get the unique grading of $\tau_{\partial M}$ which is trivial near the boundary.

Theorem 1.1. *Let $\mu : M \to M$ be the monodromy of the Lefschetz fibration π around a large circle. For sufficiently small $\varepsilon > 0$, there is a canonical map*

$$\rho : HF^{*+2}(\mu, \varepsilon) \to H^*(\hom_{(\mathcal{A},\mathcal{A})}(\mathcal{A}^\vee[-n], \mathcal{A})). \tag{58}$$

When π is a Lefschetz fibration which comes from an anticanonical Lefschetz pencil on $\overline{E}$, the Reeb flow on ∂M is periodic (say with period 1), and the monodromy is the boundary twist associated to that periodic flow:

$$\mu = \tau_{\partial M}. \tag{59}$$

This satisfies

$$HF^{*+2}(\tau_{\partial M},\varepsilon)=\begin{cases} H^*(M,\partial M) & \varepsilon\in(0,1),\\ H^*(M) & \varepsilon\in(1,2).\end{cases} \tag{60}$$

Theorem 1.1 applies only to the case when $\varepsilon\in(0,1)$. However, by a more precise analysis, one can show that in the second case, an analogue of the map ρ can be defined in the lowest degree $*=0$. Hence, the element of $HF^2(\tau_{\partial M},\varepsilon),\varepsilon\in(1,2)$, corresponding to $1\in H^0(M)$ still gives rise to a map in $H^0(\hom_{(\mathcal{A},\mathcal{A})}(\mathcal{A}^\vee[-n],\mathcal{A}))$, which we denote by σ.

2 Deformation of the Fukaya category

2.1 One-parameter formal deformations

Let $\mathfrak{g}$ be a dg Lie algebra over $\mathbb{C}$. One considers solutions $\alpha_q\in q\mathfrak{g}^1[[q]]$ (which will be referred to as *Maurer-Cartan elements*) of the Maurer-Cartan equation

$$d\alpha_q+\frac{1}{2}[\alpha_q,\alpha_q]=0. \tag{61}$$

Take the group G_q associated to the Lie algebra $q\mathfrak{g}^0[[q]]$. We denote its elements by $\exp(\gamma_q)$, for $\gamma_q\in q\mathfrak{g}^0[[q]]$. G_q acts on the set of Maurer-Cartan elements by gauge transformations

$$\exp(\gamma_q)(\alpha_q)=\alpha_q+([\gamma_q,\alpha_q]-d\gamma_q)+O(\gamma_q^2). \tag{62}$$

We say that α_q is trivial if it is gauge equivalent to 0. If we write $\alpha_q=q\alpha_1+q^2\alpha_2+\cdots$, then it follows from (61) that $d\alpha_1=0$, and the class

$$[\alpha_1]\in H^1(\mathfrak{g}) \tag{63}$$

is gauge invariant. There are standard rigidity and unobstructedness results:

Lemma 2.1. *Suppose that $H^1(\mathfrak{g})=0$, then any Maurer-Cartan element is trivial.*

Lemma 2.2. *Suppose that $H^2(\mathfrak{g})=0$. Then any class in $H^1(\mathfrak{g})$ can be realized as the first order piece α_1 of an Maurer-Cartan element α_q.*

One can use the Maurer-Cartan element to deform the differential on $\mathfrak{g}[[q]]$, namely

$$d_{\alpha_q}=d+[\alpha_q,\cdot]. \tag{64}$$

The *Kaledin class* associated to α_q is a cohomology class of this twsited cohomology theory:

$$\kappa(\alpha_q)=[\partial_q\alpha_q]\in H^1(\mathfrak{g}[[q]],d_{\alpha_q}). \tag{65}$$

The adjoint action of $\exp(\gamma)_q\in G_q$ relates d_{α_q} and $d_{\exp(\gamma_q)(\alpha_q)}$, and induces isomorphisms

$$H^*(\mathfrak{g}[[q]],d_{\alpha_q})\cong H^*(\mathfrak{g}[[q]],d_{\exp(\gamma_q)(\alpha_q)}) \tag{66}$$

which preserve the Kaledin class $\kappa(\alpha_q)$. Thus the Kaledin class is an invariant of the gauge invariant class of α_q. The following lemma shows that it detects the triviality of the Maurer-Cartan element.

Lemma 2.3. *A Maurer-Cartan element is trivial if and only if its Kaledin class vanishes.*

When applying the general theory recalled above to the specific setting of the deformation of a (curved) A_∞-algebra $\mathcal{A}$, the dg Lie algebra

$$\mathfrak{g}=CC^*(\mathcal{A},\mathcal{A})[1]=\prod_{d\geq 0}\hom^{*+1-d}(\mathcal{A}^{\otimes d},\mathcal{A}) \tag{67}$$

is defined to be the (shifted) Hochschild cochain complex, with the Lie bracket given by the Gerstenhaber bracket. The complex $(CC^*(\mathcal{A},\mathcal{A})[[q]],d_{\alpha_q})$ is the Hochschid complex of the deformed (curved) A_∞-structure $\mathcal{A}_q$.

2.2 The fundamental assumption

Within Gromov-Witten theory, the class $-\partial_q[\omega_{\overline{E},q,\overline{b}}]$ plays a special role (since it's related to the Kaledin class of our formal deformation $\mathcal{A}_{q,\bar{b}}$ via the closed-open string map). We want to consider a situation in which that class (plus a multiple of $c_1(\overline{E})$) can itself be written as a Gromov-Witten invariant.

Assumption 2.1. *There are functions $\psi(q) \in 1 + q\mathbb{C}[[q]]$ and $\eta(q) \in \mathbb{C}[[q]]$, such that*

$$-\partial_q[\omega_{\overline{E},q,\overline{b}}] = q^{-1}[\delta \overline{E}] + (\partial_q \overline{b})/\overline{b} = \psi(q) z^{(1)}_{q,\overline{b}} - \eta(q)[\overline{M}]. \tag{68}$$

(68) can be thought of as an equation for the triple $(\overline{b}, \psi, \eta)$. It admits a large group of symmetries, generated by transformations

$$(\overline{b}_\alpha, \psi_\alpha, \eta_\alpha) = (\overline{b} \cdot \alpha(q)^{[\overline{M}]}, \alpha(q)^{-1}, \psi(q), \eta(q) - \alpha(q)^{-1}\alpha'(q)), \alpha \in 1 + q\mathbb{C}[[q]], \tag{69}$$

and

$$(\overline{b}_\beta, \psi_\beta, \eta_\beta) = (\overline{b}(\beta(q)/q)^{\delta \overline{E}}, \psi(\beta(q))\beta'(q), \eta(\beta(q))\beta'(q)), \beta \in q + q^2\mathbb{C}[[q]]. \tag{70}$$

Note that by using (69), one can always reduce to the case where $\psi = 1$. After that reduction, the remaining symmetry (70) has the form

$$(\overline{b}_\beta, \eta_\beta) = (\overline{b}(\beta(q)/q)^{\delta \overline{E}} \beta'(q)^{[\overline{M}]}, \eta(\beta(q))\beta'(q) - \beta'(q)^{-1}\beta''(q)) \tag{71}$$

An order-by-order analysis of (71) shows that there is always a unique choice of such a transformation which reduces the situation to $\eta = 0$ (this fact is crucial to the proof).

Lemma 2.4. *(68) always has a solution, which moreover is unique up to the symmetries (69) and (70).*

Proof. An inductive argument. □

We finish these preliminary considerations by looking at an important special case, namely (68) with trivial bulk term.

Lemma 2.5. *Consider diffeomorphisms of $\overline{E}$ which preserve the deformation class of the symplectic form, and which map $[\delta \overline{E}]$ to itself; they also automatically preserve $[\overline{M}] = c_1(\overline{E})$. Let*

$$H^*(\overline{E})^{inv} \subset H^*(\overline{E}) \tag{72}$$

be the subspace on which all such diffeomorphisms act trivially. If that subspace is spanned by $[\delta \overline{E}]$ and $[\overline{M}]$, there is a solution of (68) with $\overline{b} = 1$.

Proof. The solvability of (68) with $\overline{b}$ is equivalent to the statement that

$$z^{(1)}_q \in q^{-1}\mathbb{C}[[q]] \cdot [\delta \overline{E}] \oplus \mathbb{C}[[q]] \cdot [\overline{M}] \subset H^2(\overline{E}) \otimes q^{-1}\mathbb{C}[[q]]. \tag{73}$$

Because Gromov-Witten invariants are symplectic deformation invariants, $z^{(1)}_q$ lies in $H^*(\overline{E})^{inv}$. □

2.3 The main theorem

Let's return to the Fukaya category $\mathcal{A}$ and its deformation $\mathcal{A}_{q,\bar{b}}$. The main theorem is:

Theorem 2.1. *Suppose that $\bar{b}$ is the restriction of some $\overline{b}$ satisfying Assumption 2.1. Then $\mathcal{A}_{q,\bar{b}}$ is a trivial deformation.*

At first, we will work more generally with Lefschetz fibrations which have fibrewise compactifications $\bar{\pi} : \overline{E} \to \mathbb{C}$. As a general feature of deformation theory, $\mathcal{A}_{q,\bar{b}}$ comes with a canonical cohomology class, the Kaledin class

$$[\partial_q \mu^*_{\mathcal{A}_{q,\bar{b}}}] \in HH^2(\mathcal{A}_{q,\bar{b}}, \mathcal{A}_{q,\bar{b}}). \tag{74}$$

Applying Lemma 2.3 implies the following:

Lemma 2.6. *$\mathcal{A}_{q,\bar{b}}$ is a trivial A_∞-deformation if and only if $[\partial_q \mu^*_{\mathcal{A}_{q,\bar{b}}}]$ vanishes.*

The next step is to replace the algebraic class $[\partial_q \mu^*_{\mathcal{A}_{q,\bar{b}}}]$ with a geometric one. Write

$$H^*(\overline{E})_q = q^{-1} H^*(\overline{E}, E) \oplus H^*(\overline{E})[[q]]. \tag{75}$$

(This definition is motivated by the expression of $[\partial_q \omega_{\overline{E},q,\bar{b}}]$, recall that $-\partial_q[\omega_{\overline{E},q,\bar{b}}] = q^{-1}[\delta E] + (\partial_q \bar{b})/\bar{b}$.) This is a graded $\mathbb{C}[[q]]$-module, where multiplication by q maps the first factor to the second one (by the standard map $H^*(\overline{E}, E) \to H^*(\overline{E})$). In particular, we have a distinguished class

$$q^{-1}[\delta E] \in q^{-1} H^2(\overline{E}, E) \subset H^2(\overline{E})_q. \tag{76}$$

We need $H^*(\overline{E})_q$ for a suitable version of the closed-open string map, which has the form

$$CO_{q,\bar{b}} : H^*(\overline{E})_q \to HH^*(\mathcal{A}_{q,\bar{b}}, \mathcal{A}_{q,\bar{b}}). \tag{77}$$

Lemma 2.7. *The Kaledin class can be expressed as follows:*

$$[\partial_q \mu^*_{\mathcal{A}_{q,\bar{b}}}] = CO_{q,\bar{b}}([-\partial_q \omega_{\overline{E},q,\bar{b}}]). \tag{78}$$

The map (77) is an analogue of the standard closed-open string map

$$CO : H^*(E) \to HH^*(\mathcal{A}, \mathcal{A}). \tag{79}$$

There is a commutative diagram

$$\begin{array}{ccc} H^*(E) & \longrightarrow & HF^*(\phi) \\ \downarrow{\scriptstyle CO} & & \downarrow{\scriptstyle \rho} \\ HH^*(\mathcal{A}, \mathcal{A}) & \xrightarrow{\cong} & H^*(\hom_{(\mathcal{A},\mathcal{A})}(\mathcal{A}, \mathcal{P})) \end{array} \tag{80}$$

(The bimodule $\mathcal{P}$ can be taken to be $\mathcal{A}^\vee[-n]$ in our specific geometric set up.) Here, ϕ is a specific automorphism of E (rotation at infinity), and $HF^*(\phi)$ its fixed point Floer cohomology. On the open string side, the automorphism induces an autoequivalence of $\mathcal{F}(\pi)$, and $\mathcal{P}$ is the $\mathcal{A}$-bimodule associated to that autoequivalence. The top horizontal map in (80) is a kind of continuation map, and the same geometric mechanism gives rise to a bimodule homomorphism $\mathcal{A} \to \mathcal{P}$, which turns out to be a quasi-isomorphism. The bottom horizontal map in (80) is induced by that homomorphism, hence is an isomorphism. As a consequence, any class in $H^*(E)$ which gets mapped to 0 in $HF^*(\phi)$ must lie in the kernel of CO. Let's suppose that our Lefschetz fibration comes from anticanonical Lefschetz pencils. Then, the top horizontal map in (80) fits into a long exact sequence

$$\cdots \to H^{*-2}_c(M) \to H^*(E) \to HF^*(\phi) \to \cdots \tag{81}$$

The conclusion is that any class in $H^*(E)$ which comes from $H^{*-2}_c(M)$ lies in the kernel of the closed-open string map.

Now consider what happens under fiberwise compactification. There is an extension of ϕ to an automorphism $\bar{\phi}$ of $\overline{E}$. One can associate to it a *relative version* of fixed point Floer cohomology, denoted by $HF^*_{q,\bar{b}}(\bar{\phi})$, as well as a bimodule $\mathcal{P}_{q,\bar{b}}$ over $\mathcal{A}_{q,\bar{b}}$. The appropriate verion of (80) looks as follows:

$$\begin{array}{ccc} H^*(\overline{E})_q & \longrightarrow & HF^*_{q,\bar{b}}(\bar{\phi}) \\ \downarrow{\scriptstyle CO_{q,\bar{b}}} & & \downarrow{\scriptstyle \bar{\rho}} \\ HH^*(\mathcal{A}_{q,\bar{b}}, \mathcal{A}_{q,\bar{b}}) & \xrightarrow{\cong} & H^*(\hom_{(\mathcal{A}_{q,\bar{b}},\mathcal{A}_{q,\bar{b}})}(\mathcal{A}_{q,\bar{b}}, \mathcal{P}_{q,\bar{b}})) \end{array} \tag{82}$$

If the image of $[-\partial_q \omega_{\overline{E},q,\bar{b}}]$ in $HF^*_{q,\bar{b}}(\bar{\phi})$ is zero, then the deformation $\mathcal{A}_{q,\bar{b}}$ must be trivial, because of the Lemmas 2.6 and 2.7.

At this point, we re-introduce the assumption that the Lefschetz fibration comes from an anticanonical Lefschetz pencil, and also require that $\bar{b}$ is the restriction of some $\overline{b}|$. The counterpart of (81) says that the top horizontal arrow from (82) fits into a long exact sequence of graded $\mathbb{C}[[q]]$-modules

$$\cdots \to H^{*-2}(\overline{M})[[q]] \to H^*(\overline{E})_q \to HF^*_{q,\bar{b}}(\bar{\phi}) \to \cdots \tag{83}$$

Let's introduce an analogue of $H^*(\overline{E})_q$,

$$H^*(\overline{E}|)_q = q^{-1}H^*(\overline{E}|, E|) \oplus H^*(\overline{E}|)[[q]]. \tag{84}$$

By (43), $z^{(1)}_{q,\overline{b}|}$ admits a natural lift to $H^2(\overline{E}|)_q$ (recall that $z^{(1)}_{q,\overline{b}|} \in q^{-1}[\delta E|] + H^2(\overline{E}|)[[q]]$), for which we use the same notation.

Lemma 2.8. *The restriction of $z^{(1)}_{q,\overline{b}|}$ to $H^2(\overline{E})_q$ equals the image of $1 \in H^0(\overline{M})[[q]]$ under the first map in (83).*

Hence, the image of $\psi(q) \in H^0(\overline{M})[[q]]$ is $\psi(q)z^{(1)}_{q,\overline{b}|}$. If one can achieve that this equals $q^{-1}[\delta E] + \partial_q \bar{b}/\bar{b}$ (which is just $[-\partial_q \omega_{\overline{E},q,\bar{b}}]$) (this follows from Assumption 2.1 and the fact that one can always achieve that $\eta = 0$), then it follows that the latter class maps to zero in $HF^*_{q,\bar{b}}(\bar{\phi})$ (since it comes from $H^0(\overline{M})[[q]]$). This explains Theorem 2.1.

2.4 Monodromy considerations

Start with a Lefschetz pencil on a del Pezzo surface $\overline{E}\rangle$. Blowing up its base locus yields a rational elliptic surface $\overline{E}|$ (topologically, $\mathbb{CP}^2$ blowing up 9 points). Let's write

$$H_2(\overline{E}|) = \mathbb{Z}L \oplus \mathbb{Z}A_0 \oplus \cdots \mathbb{Z}A_8, \tag{85}$$

with the intersection form $\mathrm{diag}(1, -1, \cdots, -1)$. One can choose the basis so that

$$[\overline{M}] = 3L - A_0 - \cdots - A_8, \tag{86}$$

$$[\delta E|] = \begin{cases} A_0 + \cdots + A_{d-1} & \text{if } \overline{E}\rangle \text{ is } \mathbb{CP}^2 \text{ blown up at } 9-d \text{ points,} \\ A_0 + \cdots + A_6 + (L - A_7 - A_8) & \text{if } \overline{E}\rangle \cong \mathbb{CP}^1 \times \mathbb{CP}^1. \end{cases} \tag{87}$$

Generally, we write d for the number of connected components of $\delta E|$.

Lemma 2.9. *There are diffeomorphisms of $\overline{E}|$ which preserve the deformation class of the symplectic form, and which realize the following automorphisms of homology:*

(1) any permutation of $\{A_0, \cdots, A_8\}$;

(2) the reflection

$$X \mapsto X + (X \cdot S), \text{ where } S = L - A_6 - A_7 - A_8. \tag{88}$$

Proof. For (i), one deforms the symplectic form so that $\overline{E}|$ is a blowup of $\mathbb{CP}^2$ whose exceptional divisors have very small area. Then, a symplectic automorphism of $\mathbb{CP}^2$ which permutes the blowup points, and permutes local Darboux charts near those points, can be lifted to an automorphism of $\overline{E}|$. For (ii), one deforms the blowup points so that the last 3 are collinear. Then, the blowup contains a (-2) curve in class S, which one can contract to an ordinary double point singularity. The diffeomorphism we are looking for is the associated monodromy map (a Dehn twist). □

Lemma 2.10. *Suppose that our Lefschetz fibration arises from a Lefschetz pencil on a del Pezzo surface which is not $\mathbb{CP}^2$ blown up at one or two points. Then $\overline{b}| = 1$ is a solution of (68).*

Proof. This follows from the above lemma and Lemma 2.5. □

It's an interesting question how the above result is related to the non-existence of Kähler-Einstein metrics on $\mathbb{CP}^2$ blown up at 1 or 2 points.

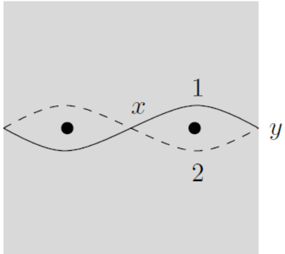

Figure 2: The vanishing cycles V_1 and V_2

3 Examples

3.1 A non-example

Let's begin with an extremely simple "made up" example. This does not come from a Lefschetz pencil, hence our main results don't apply to it.

Consider an exact Lefschetz fibration whose fibre M is a two-punctured torus, and which has a basis of vanishing cycles (V_1, V_2) drawn in Figure 2. This describes the Lefschetz fibration uniquely, up to deformation. It's easy to see that

$$\hom_{\mathcal{A}}(L_i, L_j) = \begin{cases} \mathbb{C} \cdot e_{L_i} & i = j \\ CF^*(V_1, V_2) = \mathbb{C} \cdot x + \mathbb{C} \cdot y & (i,j) = (1,2) \\ 0 & (i,j) = (2,1) \end{cases} \tag{89}$$

The degrees are $|x| = 0$, $|y| = 1$. All A_∞-operations $\mu^d_{\mathcal{A}}$ vanish. Let's consider the obvious fiberwise compactification, obtained by filling in the punctures, and use Lemma 1.1 to determine $\mathcal{A}_q$. In this case, $\mathcal{A}_q$ is the trivial deformation: we have two holomorphic bigons, whose contributions to the differential $\mu^1_{\mathcal{A}_q}(x)$ are $\pm qy$, hence cancel.

Remark. The partial compactification $\overline{E}$ contains a Lagrangian sphere, which gives rise to a spherical object in $\mathcal{F}_q(\bar{\pi})$. Algebraically, this can be written as the cone of a morphism $L_1 \to L_2$, hence lies in the derived category $D^b(\mathcal{A}_q)$. Because of the deformation is trivial, the same object appears already in $D^b(\mathcal{A}) \cong D^b(\mathcal{F}(\pi))$, in spite of the fact that there is no Lagrangian sphere in E, since $H_2(E) = 0$.

Let's consider the effect of introducing a bulk term, which can be written as

$$\bar{b} = (\beta_0)^{A_0} (\beta_1)^{A_1}, \tag{90}$$

where A_0, A_1 are the Poincaré duals of two sections of $\bar{\pi}$ corresponding to the punctures in the fiber M; and $\beta_0, \beta_1 \in 1 + q\mathbb{C}[[q]]$. The effect on our previous computation is that holomorphic polygons in the fiber $\overline{M}$ that pass over the punctures with multiplicities (m_0, m_1) should be counted with an additional weight $\beta_0^{m_0}\beta_1^{m_1}$. This means that

$$\mu^1_{\mathcal{A}_{q,\bar{b}}}(x) = q(\beta_0 - \beta_1)y. \tag{91}$$

For $\beta_0 = \beta_1$, the deformation is trivial; for $\beta_0 \neq \beta_1$, the deformation is non-trivial.

Remark. If $\beta_0 \neq \beta_1$, the Lagrangian sphere from the remark above is no longer an object of $\mathcal{F}_{q,\bar{b}}(\bar{\pi})$, since the integral of $\bar{b}$ over it is non-zero. This is an instance of the general trick which excludes unwanted objects in the Fukaya category bu turning on the bulk term. This trick is used by Smith to exclude unobstructed Lagrangian cylinders when trying to identify the Fukaya category of certain quasi-projective Calabi-Yau threefolds with quiver algebras defined by a marked bordered Riemann surface.

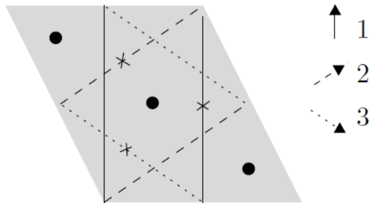

Figure 3: A basis of vanishing cycles for the mirror of $\mathbb{CP}^2$

Remark. Suppose that we modify the geometric setup by putting another puncture close to one of the two existing ones. This affects the way in which holomorphic bigons are counted, resulting in

$$\mu^1_{\mathcal{A}_q}(x) = (q - q^2)y, \tag{92}$$

which is clearly a non-trivial deformation. In this situation, there is no longer a Lagrangian sphere in $\overline{E}$, since the symplectic form will have non-zero integral over the relevant homology class. Moreover, $\mathcal{A}_{q,\bar{b}}$ will remain a non-trivial deformation for any choice of $\bar{b}$, since that changes the weights with which holomorphic polygons are counted only by invertible elements of $\mathbb{C}[[q]]$.

3.2 The mirror of $\mathbb{CP}^2$

Consider the toric mirror of the projective plane, the basis of vanishing cycles is shown in Figure 3. Within mirror symmetry, this reflects the fact that the fibrewise compactification is again a mirror of $\mathbb{CP}^2$, but with respect to a non-toric anticanonical divisor.

Let's start with the exact fibration. Since there are three vanishing cycles, and all the morphisms between them are concentrated in degree 0, the only non-trivial part of the A_∞-structure is the product

$$\mu^2_{\mathcal{A}} : CF^*(V_2, V_3) \otimes CF^*(V_1, V_2) \to CF^*(V_1, V_3). \tag{93}$$

The intersections $V_i \cap V_j$ $(i \neq j)$ each consist of 3 points, so (93) has 27 coefficients. All but 6 of these coefficients vanish, and the remaining ones are ± 1 (corresponding to the 6 small triangles in Figure 3). To get the signs right, it is important to remember that all vanishing cycles must be equipped with local systems having holonomy -1, or equivalently, with nontrivial *Spin* structures. Let's think of these local systems (or *Spin* structures) as being trivial away from one marked point on each V_k; the specific points we use is shown in Figure 3. The effect is that a triangle is counted with $(-1)^s$ if its boundary crosses the marked points s times. With respect to bases $\{x_k\}$, $\{y_k\}$, $\{z_k\}$ of the spaces in (93) given by intersection points, one then has

$$\begin{array}{lll} \mu^2_{\mathcal{A}}(y_2, x_1) = z_3, & \mu^2_{\mathcal{A}}(y_1, x_2) = -z_3, & \mu^2_{\mathcal{A}}(y_3, x_2) = z_1, \\ \mu^2_{\mathcal{A}}(y_2, x_3) = -z_1, & \mu^2_{\mathcal{A}}(y_1, x_3) = z_2, & \mu^2_{\mathcal{A}}(y_3, x_1) = -z_2, \\ \mu^2_{\mathcal{A}}(y_1, x_1) = 0, & \mu^2_{\mathcal{A}}(y_2, x_2) = 0, & \mu^2_{\mathcal{A}}(y_3, x_3) = 0. \end{array} \tag{94}$$

The corresponding products $\mu^2_{\mathcal{A}_q}$ are obtained from those in (94) by multiplying with

$$\gamma(q) = \sum_{j=0}^{\infty} (-1)^j q^{3j(3j+1)/2} - (-1)^j q^{(3j+2)(3j+3)/2} = 1 - q^3 - q^6 + \cdots \tag{95}$$

Since the relations between these products remain the same, the deformation is trivial. Remarkably, the fact that $\mu^2_{\mathcal{A}_q}(y_k, x_k)$ vanishes does not come from the absence of holomorphic triangles, but rather from an infinite number of cancellations between their contributions, at all orders of q.

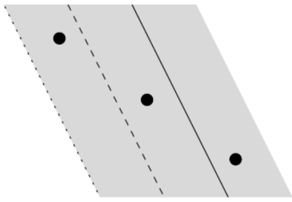

Figure 4: The additional vanishing cycles involved in the Lefschetz pencil of a cubic surface

3.3 Cubic surface

The previous example is still not a Lefschetz pencil, but one can enlarge it to one, by adding 9 more vanishing cycles: more precisely, three copies of each of the three disjoint curves shown in Figure 4 (the ordering within that group of 9 is irrelevant). The resulting basis of 12 vanishing cycles is what one gets from the anticanonical Lefschetz pencil on a cubic surface.

It turns out that the deformation $\mathcal{A}_q$ is still trivial. We give a mirror symmetry explanation.

For a complex number q, $0 < |q| < 1$, consider the following elliptic curve Y_q together with a degree 3 divisor

$$Y_q = \mathbb{C}/(\mathbb{Z} + \frac{\log(q)}{2\pi i}\mathbb{Z}), \tag{96}$$

$$Z_q = \left\{0, \frac{1}{3}, \frac{2}{3}\right\} \subset Y_q. \tag{97}$$

The associated line bundle $\mathcal{O}_{Y_q}(Z_q)$ gives rise to an embedding

$$Y_q \hookrightarrow \mathbb{P}(H^0(Y_q, \mathcal{O}_{Y_q}(Z_q))^\vee) \cong \mathbb{CP}^2. \tag{98}$$

Blow up the image of Z_q under that embedding, and lift the embedding to the blowup. Then, carry out the same process two more times. The outcome is a rational elliptic surface X_q, into which Y_q is embedded with self-intersection 0 (it has self-intersection number 9 before blowing up). $D^b Coh(X_q)$ carries a full exceptional collection with 12 objects, whose restriction to Y_q is mirror to our basis of vanishing cycles on the (compact) torus with symplectic area $-\log(q)$. In particular, if we temporarily pretend that q can be treated as a complex number on the symplectic side, we would have

$$D^b\mathcal{F}_q(\bar{\pi}) \cong D^b Coh(X_q). \tag{99}$$

The key observation is that $X = X_q$ is actually independent of q. By (98), the image of Z_q consists of three collinear points in $\mathbb{CP}^2$ (the left side of Figure 5 shows the image of Y_q, and the line through the image of Z_q; the right side shows what happens after repeatedly blowing up). Specifically for our choice (97), we have linear equivalences

$$Z_q \sim 3 \cdot \{0\} \sim 3 \cdot \{1/3\} \sim 3 \cdot \{2/3\}. \tag{100}$$

This means that for each point of Z_q, there is a section of $\mathcal{O}_{Y_q}(Z_q)$ which vanishes to third order at that point. If we use those sections as a basis of our linear system, the image of Y_q under (98) will intersect the coordinate lines at the points of Z_q, and each time with intersection multiplicity 3. As a consequence, the repeated blowups used to form X_q are carried out on the proper transforms of the coordinate lines, hence are the same for all q.

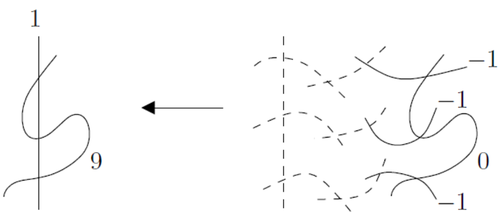

Figure 5: Configuration of curves in the rational elliptic surface X_q

Remark As one sees from Figure 5, X contains an $\widetilde{E}_6$-configuration of (-2)-curves. The mirror is a configuration of 8 Lagrangian spheres in the affine cubic surface, which is a basis of vanishing cycles for the $T_{3,3,3}$ singularity. X also contains another $\widetilde{A}_2$ configuration, disjoint from the previous one, namely the proper transforms of the coordinate lines (if there are mirror spheres to those, they need to lie in $\overline{E}$, but for topological reasons it's not possible). The existence of those two singular elliptic fibres (one $\widetilde{E}_6$ fiber and one $\widetilde{A}_2$ fiber) characterizes X uniquely. It's the extremal elliptic surface X_{431}. (*Extremal* means that the rank of $NS(X)$ equals $h^{1,1}(X)$.)